Underdetermined Blind Source Separation Using Sparse Coding

Liangli Zhen, Dezhong Peng, Zhang Yi, *Fellow, IEEE*, Yong Xiang, *Senior Member, IEEE*, and Peng Chen

***Abstract*— In an underdetermined mixture system with n unknown sources, it is a challenging task to separate these sources from their m observed mixture signals, where $m < n$. By exploiting the technique of sparse coding, we propose an effective approach to discover some 1-D subspaces from the set consisting of all the time-frequency (TF) representation vectors of observed mixture signals. We show that these 1-D subspaces are associated with TF points where only single source possesses dominant energy. By grouping the vectors in these subspaces via hierarchical clustering algorithm, we obtain the estimation of the mixing matrix. Finally, the source signals could be recovered by solving a series of least squares problems. Since the sparse coding strategy considers the linear representation relations among all the TF representation vectors of mixing signals, the proposed algorithm can provide an accurate estimation of the mixing matrix and is robust to the noises compared with the existing underdetermined blind source separation approaches. Theoretical analysis and experimental results demonstrate the effectiveness of the proposed method.**

***Index Terms*— Mixing matrix identification, single source detection, source recovery, sparse coding, underdetermined blind source separation (UBSS).**

I. Introduction

Blind source separation (BSS) aims at separating the original source signals from their mixtures without any *a priori* knowledge about the mixing matrix and the source signals [1]–[3]. BSS has attracted considerable research attentions because of its wide applications in biomedical engineering, remote sensing, speech recognition, and communication systems [4]. A large number of BSS algorithms have been developed in the past decades, and most of them assume that the number of sensors is not less than the number of sources. In practice, however, this assumption is difficult to be satisfied. For example, in a wireless sensor network, the number of sources is sometimes unknown to the receivers. Thus, the number of the disposed receivers could be less than the number of the sources, which leads to a more challenging problem, i.e., underdetermined BSS (UBSS) [5]–[13].

To solve the above UBSS problem, many methods have been proposed by exploiting the sparsity of sources in time domain [5] or time-frequency (TF) domain [3], [6]–[10], [14]–[25]. In the early works based on TF representations [7], [14], the source signals are assumed to be TF-disjoint, i.e., there exists at most one active source at any point in the TF domain. For example, using linear TF distributions (TFDs) [7]–[10], Jourjine *et al*. [7] proposed the degenerate unmixing estimation technique (DUET) to recover the source signals based on the ratios of the TF transforms of the observed mixing signals. Linh-Trung *et al*. [14] proposed to exploit quadratic TFDs [3], [20] to separate the sources. It is notable that this TF-disjoint constraint is restrictive and easy to be violated in the real-world applications [15]. Although the methods in [7] and [14] can still work when there are some small overlapping among different sources in the TF domain, the separation performance will degrade as the number of these overlapping TF points is increasing [15]. In order to relax this TF-disjoint constraint, the time-frequency ratio of mixtures (TIFROM) algorithm is proposed in [8], which allows overlapping of source TF representations to a certain degree, but still needs some adjacent TF windows where only one source occurs.

In recent years, based on a more relaxed sparsity assumption that for each source, there are some TF points where only single source occurs, many methods have been proposed for UBSS. They detect the TF points where only one source is active. These points are referred to as single source points (SSPs) [9], [21]. The mixture vectors at these SSPs will be clustered to estimate the mixing matrix. Thirion-Moreau [20] and Linh-Trung *et al.* [18] suggested to detect rank-1 matrices to estimate the mixing matrix. Li *et al.* [10] proposed to detect directly several submatrices of the ratio matrix each of which has almost identical columns to estimate the mixing matrix. Reju *et al.* [9] proposed to identify the SSPs by comparing the absolute directions of the real and imaginary parts of the Fourier transform coefficient vectors of the observable mixtures. It treats one specific TF point as an SSP if the difference between the absolute directions of the real part and image part of the mixture vector at this TF point is less than a given threshold angle.

It is notable that the performance of all the above-mentioned UBSS algorithms [7]–[10], [14], [15] depends highly on the correct detection of SSPs. These methods are all based on the ratios between different mixture signals at each TF point to achieve the estimation for the mixture matrix. Clearly, these ratios are sensitive to noises, which result in a performance degradation of these UBSS algorithms in noisy environments.

To overcome the above problem, this brief proposes a novel UBSS method to separate n sources from their $m(m < n)$ instantaneous mixtures, which exploits the sparse coding of TF representation vectors of the observed mixture signals to achieve the UBSS. More specifically, we code each TF vector as a sparse linear combination of other mixture TF vectors. By enforcing ℓ_1-regularization on the coding coefficient vector, the sparse coding strategy prefers to select a few mixture TF vectors from the same subspace as the target vector to reconstruct it. Based on the obtained sparse coding coefficients, we can identify the mixture TF vectors at SSPs, which lie in n different 1-D subspaces. These mixture TF vectors at the SSPs are further clustered to estimate the mixing matrix. After obtaining the mixing matrix estimation, we formulate the source recovery as a series of least square problems. By solving these least squares problems, the source signals can be recovered.

Our main contributions are summarized as follows. We propose a novel and effective method to detect the mixture TF vectors at SSPs, where only one source occurs, in the TF domain. It exploits sparse

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coding to detect the mixture TF vectors at SSPs. Since the sparse coding strategy considers the overall linear representation relations among the TF vectors of mixing signals, our proposed algorithm can provide a more accurate estimation for the mixing matrix than the existing mixing matrix estimation approaches in [7]–[9] and [14], which detect the mixture TF vectors at SSPs by exploiting the ratios of mixing signals.

The rest of this brief is organized as follows. Section II presents the new UBSS algorithm. Section III provides experimental results to illustrate the effectiveness of the proposed algorithm. Section IV concludes this brief.

Notations: In this brief, lower-case bold letters represent column vectors, upper-case bold letters represent matrices, and the entries of matrices and vectors are denoted with subscripts. For example, $\mathbf{v}$ denotes a column vector, whose ith element is v_i. For a given matrix $\mathbf{M}$, $\mathbf{m}_j$ stands for its jth column, and $\mathbf{m}'_i$ denotes its ith row. $\mathbf{M}^T$ represents the transpose of $\mathbf{M}$, whose inverse and pseudo-inverse are denoted by $\mathbf{M}^{-1}$ and $\mathbf{M}^{\dagger}$. $\mathbf{I}$ stands for the identity matrix.

II. Proposed UBSS Method

Consider the following linear instantaneous mixing system with n inputs and m outputs:

$$\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t) \tag{1}$$

where $\mathbf{s}(t) = [s_1(t), s_2(t), \ldots, s_n(t)]^T$ is an n-dimensional column vector, $s_i(t)$ denotes the sample of the ith source at t time instant, $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_m(t)]^T$ denotes an m-dimensional mixture vector, and $x_i(t)$ stands for the observed value of the ith sensor at t time instant. $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n]$ is the mixing matrix, whose ith column is called as the steering vector corresponding to the ith source.

Using short time Fourier transform (STFT) [26], the above instantaneous mixing system can be transformed into the TF domain as follows:

$$\widetilde{\mathbf{x}}(t,k) = \mathbf{A}\widetilde{\mathbf{s}}(t,k) \tag{2}$$

where $\widetilde{\mathbf{x}}(t,k) = [\widetilde{x}_1(t,k), \widetilde{x}_2(t,k), \ldots, \widetilde{x}_m(t,k)]^T$, $\widetilde{\mathbf{s}}(t,k) = [\widetilde{s}_1(t,k), \widetilde{s}_2(t,k), \ldots, \widetilde{s}_n(t,k)]^T$, $\widetilde{x}_i(t,k)$, and $\widetilde{s}_j(t,k)$ are the STFT coefficients of the ith mixture signal and the jth source signal in the kth frequency bin at t time instant, respectively.

In this section, we present a new UBSS algorithm that explores the sparsity of sources in the TF domain from the following aspects. We begin by formulating UBSS with SSP as a sparse coding problem. The mixture TF vectors at these SSPs lie in a subset of the TF vectors of the observed mixtures, which are further clustered for mixing matrix identification. Finally, we recover the source signals by solving a set of least squares problems.

It is notable that the proposed method is different from other sparse coding-based BSS methods. In [27], the aim of sparse coding is to find sparse dictionaries from the mixtures; therefore, only a small number of coefficients in the source signals are needed to encode the observed mixtures. In [28], it assumes that the sources can be sparsely represented in one particular orthonormal basis. Each source and its corresponding column of the mixing matrix are alternately estimated in an iterative manner based on sparse coding.

In the following, we will present the new UBSS method in detail.

A. Mixing Matrix Identification

To identify the mixing matrix, we propose the following two assumptions on the mixing matrix and the sparsity of sources.

Assumption 1: For each source $\mathbf{s}'_i$, there are some TF points (t,k) where only $\mathbf{s}'_i$ is dominant, i.e., $|\widetilde{s}_i(t,k)| \gg |\widetilde{s}_j(t,k)|, \forall j \neq i$.

Assumption 2: Any m column vectors in the mixing matrix $\mathbf{A}$ are linearly independent.

The Assumption 1 is much more relaxed than the constraints adopted by DUET [7] and TIFROM [8]. More specifically, DUET assumes that at most one source is dominant at each TF point, and TIFROM assumes that there exist some adjacent TF windows where only one source occurs. Clearly, the assumption adopted by our method only needs the existence of some TF points where only one source is dominant. In other words, these SSPs are allowed arbitrarily distributed in the TF plane. This constraint is called as SSP condition in [21]. The Assumption 2 guarantees that all the sources can be recovered. Clearly, this assumption is satisfied with probability one for any randomly generated mixing matrix. It is a widely used assumption in the recent UBSS algorithms [5], [9], [10], [29].

Based on the above assumptions, for a TF point (μ, ν) on which only one source $\mathbf{s}'_i$ is active, we have

$$\widetilde{\mathbf{x}}(\mu,\nu) = \widetilde{s}_i(\mu,\nu)\mathbf{a}_i. \tag{3}$$

It means that the ith column vector $\mathbf{a}_i$ in the mixing matrix equals to the observed mixture TF vector $\widetilde{\mathbf{x}}(\mu,\nu)$ up to a multiplicative coefficient at this SSP (μ,ν). Clearly, we can obtain the estimations for the column vectors in the mixing matrix by exploiting the mixture vectors at these SSPs.

Another key observation is that mixture TF vectors with the same single active source lie in an 1-D subspace, and these mixture TF vectors can be linearly represented by another mixture TF vector in this subspace. Therefore, if $\widetilde{\mathbf{x}}(\mu,\nu)$ and $\widetilde{\mathbf{x}}(\psi,\omega)$ are with the same single active source, there exists a real number α, such that the following condition can be satisfied:

$$\widetilde{\mathbf{x}}(\mu,\nu) = \alpha\widetilde{\mathbf{x}}(\psi,\omega). \tag{4}$$

This allows us to transform the SSPs detection problem into discovering 1-D subspaces in the sets of mixture TF vectors.

Sparse coding has been proven to be a powerful technique to discover such low-dimensional subspaces. It tries to code each mixture TF vector as a linear combination of fewest number of other mixture TF vectors. From (4), we can see that the mixture TF vector whose sparse coding solution has only one nonzero element, must lie in one of the above-mentioned 1-D subspaces with probability one.

At this point, the key procedure is how to obtain the sparse coding solution for TF representation vectors of the observed mixture signals. In the following, we introduce the solution for this problem in detail.

Let $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_Q$ be the collection of all the mixture TF vectors, where Q is the number of TF points. From the self-expressiveness property of the mixture TF vectors, we can code each mixture TF vector as a linear combination of other mixture TF vectors, that is

$$\mathbf{y}_i = \mathbf{Y}\mathbf{c}_i, \quad \text{s.t. } c_{ii} = 0 \tag{5}$$

where $\mathbf{Y} \triangleq [\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_Q]$, $\mathbf{c}_i \triangleq [c_{i1}, c_{i2}, \ldots, c_{iQ}]$ is the coding coefficient vector and the constraint $c_{ii} = 0$ eliminates the trivial solution of representing a mixture TF vector by itself.

Sparse coding tries to get the solution, $\mathbf{c}_i$, whose nonzero entries corresponds to mixture TF vectors from the same subspace as $\mathbf{y}_i$. More specifically, a mixture TF vector $\mathbf{y}_i$ generated by the p sources can be written as a linear combination of p other mixture TF vectors produced by these same sources. As a result, ideally, the sparse coding solution only selects mixture TF vectors from the same subspace to represent $\mathbf{y}_i$.

The original sparse coding attempts to solving the following ℓ_0-norm minimization problem:

$$\min \|\mathbf{c}_i\|_0 \quad \text{s.t. } \mathbf{y}_i = \mathbf{Y}\mathbf{c}_i, \quad \mathbf{c}_{ii} = 0 \tag{6}$$

Algorithm 1 Mixing Matrix Estimation Process of Our Proposed Method

Input: The observed mixtures $\mathbf{x}(t), t = 1, \ldots, N$ with n sources, the sparsity constraint parameter λ, and $\Omega = \emptyset$.

1: Transform the observed mixtures from the time domain into the TF domain by using STFT [26].
2: Normalize all the mixture vectors to have a unit norm.
3: For each mixture TF vector, compute the sparse coding coefficients over all the mixture TF vectors by minimizing (8) via ℓ_1-Homotopy [31] MATLAB package.
4: Add the mixture TF vectors whose sparse coding coefficient vector contains only one nonzero element into Ω.
5: Apply clustering method on Ω to group its elements into n clusters.
6: Calculate the centers of these n clusters as the estimations for the columns of the mixing matrix .

Output: The estimated mixing matrix.

where $\|\mathbf{c}_i\|_0$ is the ℓ_0-norm and is equivalent to the number of nonzero elements in the vector $\mathbf{c}_i$. It is a general NP-hard problem [30] due to its nature of combinational optimization.

Recent development in the theory of compressed sensing reveals that if the solution of $\mathbf{c}_i$ is sufficiently sparse, the solution of problem (6) is equal to the solution of the following ℓ_1-norm minimization problem [30]:

$$\min \|\mathbf{c}_i\|_1 \quad \text{s.t. } \mathbf{y}_i = \mathbf{Y}\mathbf{c}_i, \quad \mathbf{c}_{ii} = 0 \tag{7}$$

where $\|\mathbf{c}_i\|_1$ is the ℓ_1-norm of $\mathbf{c}_i$. This problem can be solved in polynomial time by the standard linear programming methods [30].

Moreover, the mixture procedures of source signals are usually corrupted by the noises. To improve the robustness of our method, we introduce a construction error term and formulate the sparse coding problem as follows:

$$J(\mathbf{c}_i; \lambda) = \lambda\|\mathbf{c}_i\|_1 + \frac{1}{2}\|\mathbf{y}_i - \mathbf{Y}\mathbf{c}_i\|_2^2 \quad \text{s.t. } \mathbf{c}_{ii} = 0 \tag{8}$$

where $\lambda > 0$ is a scalar regularization parameter that balances the tradeoff between sparsity and reconstruction error. As this is a convex optimization problem, we can solve it efficiently using the convex optimization methods [30]. In this brief, we use the ℓ_1-norm solver that is from MATLAB package: ℓ_1-Homotopy [31], to calculate the optimal solution of (8). The homotopy method recovers solutions with p nonzeros in $O(p^3 + Q)$ time, where Q is the number of the columns in $\mathbf{Y}$ [31].

Once getting the sparse coding solutions at each TF point, we treat the TF points where the sparse coding coefficient vector has only one nonzero element as the SPPs. By clustering the mixture TF vectors at these detected SSPs, we can obtain the estimation for the mixing matrix. More specifically, we can cluster these single source mixture TF vectors by some robust clustering methods [9], [12], [32], [33] into n groups, and take the center of each group as a steering vector in the mixing matrix. The value of n is given as a parameter or learned from the mixtures by some other methods.

Based on the above analysis, we propose the following Algorithm 1 to estimate the mixing matrix.

B. Source Recovery

After the mixing matrix is estimated, the source recovery is not a trivial task, since the mixing matrix is irreversible in the UBSS problem [24], [25]. In order to recover the sources, we proceed with the definition of a set $\boldsymbol{\mathcal{A}}$, and introduce another assumption on the sparsity of sources as follows:

Algorithm 2 Source Recovery Process of Our Proposed Method

Input: The observed mixtures $\mathbf{x}(t), t = 1, \ldots, N$ with n sources, and the estimated mixing matrix $\mathbf{A}$.

1: Transform the observed mixtures from the time domain into the TF domain using STFT.
2: For each mixture TF vector, find out the corresponding submatrix $\boldsymbol{\mathcal{A}}_*$ via equ. (12).
3: Calculate the estimation of source TF representation in each TF point via equ. (11).
4: Convert the estimated source TF representations back into the time domain by employing the Inverse Short Time Fourier Transform (ISTFT) [26].

Output: The recovered source signals.

Definition 1: $\boldsymbol{\mathcal{A}}$ is a set composed of all $m \times (m-1)$ submatrices of the matrix $\mathbf{A}$, that is

$$\boldsymbol{\mathcal{A}} = \{\boldsymbol{\mathcal{A}}_i | \boldsymbol{\mathcal{A}}_i = [\mathbf{a}_{\theta_1}, \mathbf{a}_{\theta_2}, \ldots, \mathbf{a}_{\theta_{m-1}}]\}. \tag{9}$$

Clearly, $\boldsymbol{\mathcal{A}}$ contains $C_n^{(m-1)}$ elements.

Assumption 3: At most $m-1$ sources among n sources are active at each TF point.

This condition is easy to satisfy in the BSS problems with speech sources, which has been used in [15]. Under Assumption 2 and Assumption 3, we have the following conclusions.

Lemma 1: For any given mixture TF vector $\widetilde{\mathbf{x}}(t, k)$, there must exist a submatrix $\boldsymbol{\mathcal{A}}_* = [\mathbf{a}_{\phi_1}, \mathbf{a}_{\phi_2}, \ldots, \mathbf{a}_{\phi_{m-1}}]$ in the set $\boldsymbol{\mathcal{A}}$, such that

$$\widetilde{\mathbf{x}}(t, k) = \boldsymbol{\mathcal{A}}_* \boldsymbol{\mathcal{A}}_*^{\dagger} \widetilde{\mathbf{x}}(t, k) \tag{10}$$

where $\boldsymbol{\mathcal{A}}_*^{\dagger}$ is the pseudo-inverse of $\boldsymbol{\mathcal{A}}_*$.

Proof: See Appendix A. ■

For each TF point (t, k), we have a corresponding $\boldsymbol{\mathcal{A}}_* = [\mathbf{a}_{\phi_1}, \mathbf{a}_{\phi_2}, \ldots, \mathbf{a}_{\phi_{m-1}}]$, which satisfies (10). Using $\boldsymbol{\mathcal{A}}_*$, we construct an n-dimensional vector $\hat{\mathbf{s}}(t, k)$ by setting its jth element as

$$\hat{s}_j(t, k) = \begin{cases} e_i, & \text{if } j = \phi_i \\ 0, & \text{otherwise} \end{cases} \tag{11}$$

where $\mathbf{e} = [e_1, e_2, \ldots, e_{m-1}]^T = \boldsymbol{\mathcal{A}}_*^{\dagger} \widetilde{\mathbf{x}}(t, k)$. Next, based on Lemma 1, we can obtain the following theorem.

Theorem 1: The constructed vector $\hat{\mathbf{s}}(t, k)$ equals to the source TF vector $\widetilde{\mathbf{s}}(t, k)$ with probability one.

Proof: See Appendix B. ■

Theorem 1 provides a theoretical foundation to estimate the TF representation $\widetilde{\mathbf{s}}(t, k)$ of the original source signals.

In noisy environments, we cannot find the submatrix $\boldsymbol{\mathcal{A}}_*$ satisfying exactly (10). Instead, we can get it from the following criterion:

$$\boldsymbol{\mathcal{A}}_* = \arg \min_{\boldsymbol{\mathcal{A}}_i \in \boldsymbol{\mathcal{A}}} \|\widetilde{\mathbf{x}}(t, k) - \boldsymbol{\mathcal{A}}_i \boldsymbol{\mathcal{A}}_i^{\dagger} \widetilde{\mathbf{x}}(t, k)\|_2. \tag{12}$$

Finally, we can convert the source TF representations obtained via Theorem 1 back into the time domain by employing the inverse STFT [26].

In summary, the source recovery process of our proposed approach is described in Algorithm 2.

III. Experiments and Results

In this section, we conduct some experiments to verify the effectiveness of the proposed algorithm. In all the experiments, the STFT size is set as 1024, time step equals to 512, and Hanning window is used as the weighting function. In all the noisy situations, 100 times of Monte Carlo simulations are conducted to evaluate

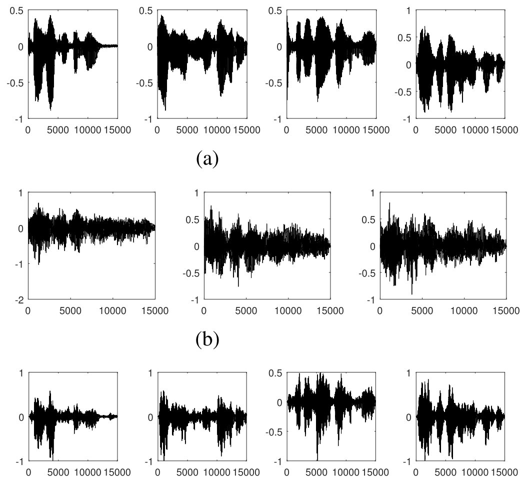

Fig. 1. (a) Source signals. (b) Mixture signals. (c) Recovered signals. Four speech signals in (a) are mixed into three mixtures in (b) and the recovered signals by the proposed method in (c).

the performance of the proposed method versus signal-to-noise ratio (SNR). For the proposed method, the mixture TF vectors at SSPs are clustered via hierarchical clustering [9].

To eliminate possible permutations of the mixing matrix and the sources, we denote the mixing matrix estimation via the UBSS method as $\bar{\mathbf{A}}$, the corresponding sources as $\bar{\mathbf{S}}$, the final mixing matrix estimation $\hat{\mathbf{A}}$, and the final estimated sources $\hat{\mathbf{S}}$. For each column in mixing matrix $\mathbf{a}_i, i = 1, 2, \ldots, n$, we find its corresponding estimation $\hat{\mathbf{a}}_i$ as the column in $\bar{\mathbf{A}}$, which has a closest absolute angle with $\mathbf{a}_i$. The source in $\bar{\mathbf{S}}$ associated with $\hat{\mathbf{a}}_i$ is the final estimation of the ith source, which is denoted as $\hat{\mathbf{s}'}_i$.

To evaluate the performance of mixing matrix estimation, we use the following performance index:

$$Error = \frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{\mathbf{a}_i^T \hat{\mathbf{a}}_i}{\|\mathbf{a}_i\|\|\hat{\mathbf{a}}_i\|}\right) \tag{13}$$

where $\hat{\mathbf{a}}_i$ denotes the estimation of the mixing steering vector $\mathbf{a}_i$, and n stands for the number of sources. To measure the performance of source recovery, the mean squared error (MSE) is used, which has the following definition:

$$\text{MSE} = 10\log_{10}\left(\frac{1}{n}\sum_{i=1}^{n}\min_{\delta}\frac{\|\mathbf{s}'_i - \delta\hat{\mathbf{s}}'_i\|_2^2}{\|\mathbf{s}'_i\|_2^2}\right) \tag{14}$$

where $\hat{\mathbf{s}}'_i$ denotes the estimation of the source signal $\mathbf{s}'_i$, and δ is a scalar reflecting the scalar ambiguity.

In the first experiment, our proposed method achieves blind separation of an underdetermined mixture system with $m = 3$ outputs and $n = 4$ input speech sources. In this system, the 3×4 mixing matrix is given by

$$\mathbf{A} = \begin{pmatrix} 0.5877 & 0.6025 & 0.5358 & 0.7078 \\ 0.4494 & -0.5525 & 0.6552 & -0.4767 \\ 0.6728 & -0.5760 & -0.5325 & 0.5212 \end{pmatrix} \tag{15}$$

and four speech sources with 15000 samples in time domain are shown in Fig. 1(a), and the observed mixing results are shown in Fig. 1(b).

By using Algorithm 1 with $\lambda = 0.001$, we get the mixing matrix estimation $\hat{\mathbf{A}}$ as shown in (16). Then, inputting the observed mixtures and the mixing matrix estimation $\hat{\mathbf{A}}$ into Algorithm 2, we can obtain the source recovery results as shown in Fig. 1(c)

$$\widehat{\mathbf{A}} = \begin{pmatrix} 0.6189 & 0.6230 & 0.5670 & 0.6927 \\ 0.4566 & -0.5428 & 0.6013 & -0.4849 \\ 0.6391 & -0.5633 & -0.5629 & 0.5339 \end{pmatrix}. \tag{16}$$

From the above results, we can find that the estimated mixing matrix $\widehat{\mathbf{A}}$ is very close to the real mixing matrix $\mathbf{A}$, and all the sources have been recovered successfully.

In the second experiment, we compare the performance of the proposed algorithm with the DUET method[1] [7], TIFROM method[2] [8], and the method[3] in [9]. We use four speech audio sources of 378035 samples with a sample rate of 16 kHz as the source signals. Moreover, additive noise signals with the normal distribution are considered. The mixing matrix is given by

$$\mathbf{A} = \begin{pmatrix} 0.6542 & 0.4811 & 0.7923 & 0.2840 \\ -0.1638 & 0.5780 & -0.1192 & 0.6192 \\ 0.7383 & -0.6591 & -0.5984 & 0.7321 \end{pmatrix}. \tag{17}$$

For the DUET method, we only input the mixtures from first two sensors, since it is designed for two mixtures separation. For the TIFROM method, the threshold of the distance between a new column of estimated mixing matrix and each previous found one is set as 0.5, which is suggested in the code provided by Abrard and Deville [8]. The parameter $\triangle\theta$ in Reju's method is set as $0.8°$ as suggested in [9]. As the TIFROM method cannot exact any source signals from the mixtures and the method in [9] only considers the mixing matrix estimation, these two methods are used to only estimate the mixing matrix. Their estimation of the mixing matrix are then inputted to the method in [5] to recover the source signals. The parameter λ of the proposed method is taken as 0.001. Please note that our method can get both the mixing matrix estimation and the separated sources. Figs. 2 and 3 show the performance comparison of the proposed algorithm and other three methods in the different noise scenarios.

From the results, we find that with the increase of SNR, the $Error$ and MSE of all the tested algorithms are decreased. Furthermore, the source recovery performance depends on the mixing

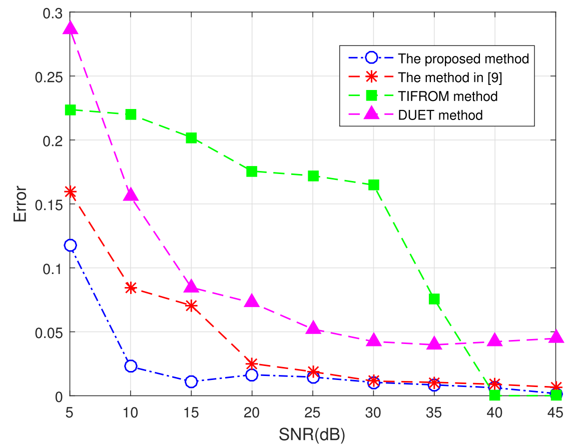

Fig. 2. Performance comparison of matrix estimation with the proposed method, DUET method, TIFROM method, and the algorithm in [9] from 378035 mixture samples.

[1] The source code of DUET method is provided by its authors via the Appendix in Chapter 8 of Blind Speech Separation, 2007, Springer.

[2] The source code of the TIFROM method is from LI-TIFROM software at http://www.ast.obs-mip.fr/article715.html

[3] The source code of the method in [9] is provided by its first author at http://www3.ntu.edu.sg/home/Reju/IBSS.zip.

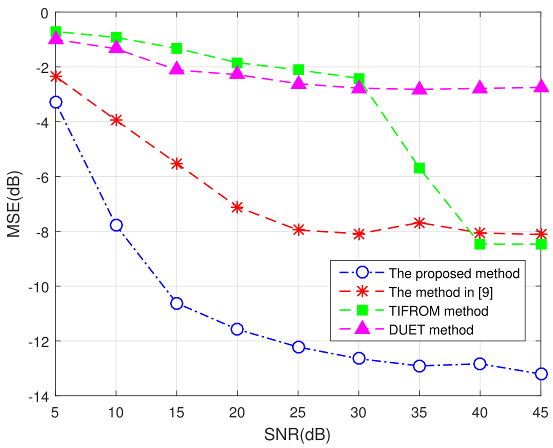

Fig. 3. Performance comparison of source recovery with the proposed method, DUET method, TIFROM method, and the algorithm in [9] from 378035 mixture samples.

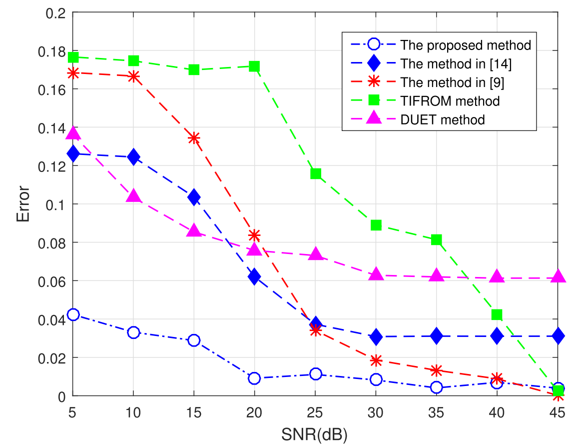

Fig. 4. Performance comparison of mixing matrix estimation with the proposed method, the method in [14], the algorithm in [9], the TIFROM method, and the DUET method from 4096 mixture samples.

TABLE I
TIME COST OF MIXING MATRIX ESTIMATION AND SOURCE RECOVERY OF DIFFERENT METHODS OVER 378035 MIXTURE SAMPLES

Metric	DUET	TIFROM	The method in [9]	Our method
$t1$	0.2281	3.0449	2.1946	3.5647
$t2$	0.5136	0.1251	0.1195	0.6214
$t3$	0.7417	3.1700	2.3141	4.1861

matrix estimation accuracy. From Fig. 2, we get that the method in [9] and the TIFROM method have much better performance than the DUET method for mixing matrix estimation when SNR $\geq$ 40 dB. The TIFROM method is more sensitive to the noise, which results in that it has a high error when SNR $\leq$ 30 dB. The method in [9] outperforms the DUET method and the TIFROM method under different noise levels, and is inferior to our proposed method in mixing matrix estimation. From Fig. 3, we observe that the proposed method outperforms the DUET method, the TIFROM method, and the method in [9] at all noisy levels. Although the method in [9] achieved low error on mixing matrix estimation when SNR $\geq$ 20 dB, the source recovery performance is much lower than that of the proposed method.

To investigate the complexity of the proposed, we suppose that there are ρ TF points in the TF plane. For each TF mixture vector, we use γ other TF mixture vectors to linearly reconstruct it. If the proposed method detected η SSPs, it takes $O(\rho(m^3+\gamma)+\eta^2)$ to estimate the mixing matrix, and $O(\rho(m^3+C_n^{m-1}))$ to recover the source signals. In the experiment, we set $\gamma = 200$, and randomly select 5000 mixture TF vectors to estimate the mixing matrix for the proposed method. The time cost of different methods with SNR = 45 dB is summarized in Table I, where $t1$ denotes the CPU elapsed time (s) for mixing matrix estimation stage, $t2$ denotes that of source recovery stage, and $t3$ is the whole time cost for the UBSS. From Table I, we find that the proposed method is slightly slower than other methods, since it considers the sparse linear representation relationships among all the mixture TF vectors. This experiment demonstrates that the proposed method can get a higher accuracy at the cost of adding a little more computational complexity.

In the third experiment, we investigate the performance of the above-mentioned methods and a quadratic TFDs-based method in [14] on mixing matrix estimation with insufficient mixture samples. We extract a patch of samples with the length of 4096 from the speech samples, which are used in the second case. Moreover, the

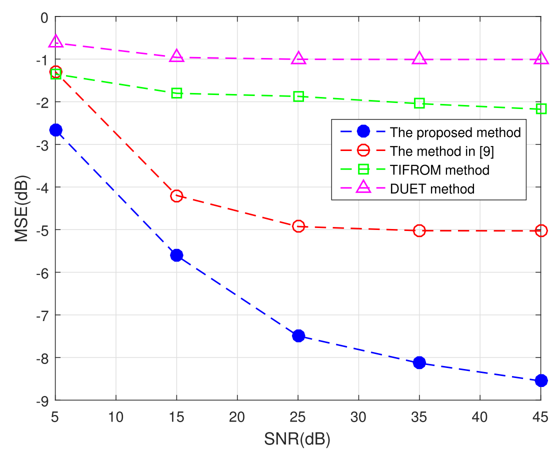

Fig. 5. Performance comparison of source separation on cocktail party problem with the proposed method, the algorithm in [9], the TIFROM method, and the DUET method.

parameters of the method in [14] is set as $\epsilon_1 = 0.05$ and $\epsilon_2 = 0.8$. The performance comparison of the mentioned methods are shown in Fig. 4. The CPU elapsed time of these 4096 samples with SNR = 45 dB for the proposed method, the method in [14], the algorithm in [9], TIFROM method, and DUET method are 1.8569, 12.8193, 1.7278, 0.11348, and 0.2191 s, respectively.

From the results, we observe that the method in [14] outperforms the DUET method. As both of them assume that the source signals are TF-disjoint, it means that the source signals are more sparse in quadratic TFDs than in the linear TFDs. The proposed method can still work well in such challenging situations and outperforms other methods, since the linear relations among the TF vectors at different TF points are considered. Moreover, the proposed method has higher efficiency than the quadratic TFDs-based method in [14].

In the fourth experiment, to evaluate the performance of the proposed method on the real-world applications, we use a public benchmark of cocktail party problem, which is provided by the ICA research center at the Helsinki University of Technology and available at http://research.ics.aalto.fi/ica/cocktail/cocktail_en.cgi. Specifically, we separate four sources from the outputs from three microphones, and compare the source estimation with its corresponding original source signals. Fig. 5 shows the separation performance of the

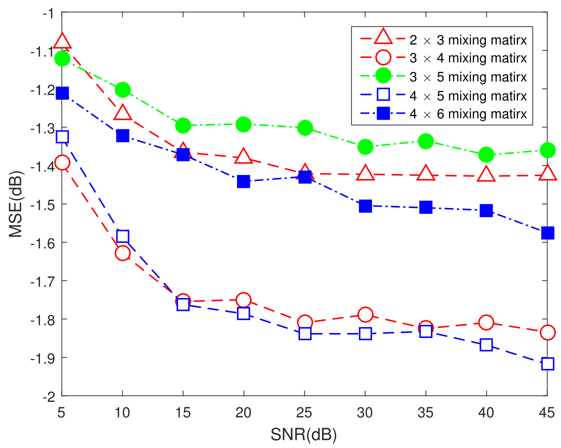

Fig. 6. Performance comparison of source recovery with different number of sources and sensors.

proposed method,[4] the DUET method, the TIFROM method, and the method in [9] under different noisy levels. From the result, we can see that the proposed method achieves the ideal performance under different noisy levels.

In the last experiment, we assess the performance of the proposed method with different number of sources and sensors. We consider the mixing matrices with the dimensions of $2 \times 3, 3 \times 4, 3 \times 5, 4 \times 5$, and 4×6, i.e., the mixture cases with three sources two sensors, four sources three sensors, five sources three sensors, five sources four sensors, and six sources four sensors, respectively. The mixing matrices and the source signals are generated randomly, and the source separation performance is shown in Fig. 6.

From the result, we find that the performance with the 3×4 mixing matrix is similar to that of the 4×5 mixing matrix, and the performance with the 3×5 mixing matrix is similar to that of the 4×6 mixing matrix. It means that the separation performance mainly depends on the difference between the number of sources and the number of sensors. However, the case with 2×3 mixing matrix is an exception. In this case, Assumption 3 requires that there exist at most one active sources at each TF point. This restriction is hard to satisfy, which results in the performance degradation.

IV. Conclusion

This brief focus on the problem of underdetermined blind separation, where the number of sensors is less than the number of sources. We exploit the sparse coding strategy among the mixture TF vectors to detect the SPPs. After detecting the SSPs in the TF plane, we get the estimation for the mixing matrix by grouping the mixture TF vectors at these SSPs. The source recovery is completed by solving a set of least squares problems. The proposed assumptions are easy to be satisfied in the practical applications. Experimental results demonstrate that the proposed method can achieve satisfactory performance.

Appendix A
Proof of Lemma 1

Proof: According to Assumption 3, at most $m-1$ sources are active at each TF point (t,k), which means that the source signal representation vector $\widetilde{\mathbf{s}}(t,k)$ contains at most $m-1$ nonzero elements. Here, we denote the indices of these $m-1$ nonzero elements as $\phi_1, \phi_2, \ldots, \phi_{m-1}$.

Obviously, there must exist a submatrix $\boldsymbol{\mathcal{A}}_* = [\mathbf{a}_{\phi_1}, \mathbf{a}_{\phi_2}, \ldots, \mathbf{a}_{\phi_{m-1}}]$ in the set $\boldsymbol{\mathcal{A}}$ satisfying

$$\mathbf{A}\widetilde{\mathbf{s}}(t,k) = \boldsymbol{\mathcal{A}}_* \widetilde{\mathbf{s}}_{\phi_1,\phi_2,\ldots,\phi_{m-1}}(t,k) \tag{18}$$

where $\widetilde{\mathbf{s}}_{\phi_1,\phi_2,\ldots,\phi_{m-1}}(t,k)$ is constructed by the elements from $\widetilde{\mathbf{s}}(t,k)$ with the indices as $\phi_1, \phi_2, \ldots, \phi_{m-1}$.

From (2) and (18), it follows that:

$$\begin{aligned}\boldsymbol{\mathcal{A}}_*\boldsymbol{\mathcal{A}}_*^\dagger \widetilde{\mathbf{x}}(t,k) &= \boldsymbol{\mathcal{A}}_*\boldsymbol{\mathcal{A}}_*^\dagger (\mathbf{A}\widetilde{\mathbf{s}}(t,k))\\ &= \boldsymbol{\mathcal{A}}_*\boldsymbol{\mathcal{A}}_*^\dagger (\boldsymbol{\mathcal{A}}_*\widetilde{\mathbf{s}}_{\phi_1,\phi_2,\ldots,\phi_{m-1}}(t,k))\\ &= \boldsymbol{\mathcal{A}}_*\widetilde{\mathbf{s}}_{\phi_1,\phi_2,\ldots,\phi_{m-1}}(t,k)\\ &= \mathbf{A}\widetilde{\mathbf{s}}(t,k)\\ &= \widetilde{\mathbf{x}}(t,k).\end{aligned} \tag{19}$$

This completes the proof. ■

Appendix B
Proof of Theorem 1

Proof: According to the number of active sources, denoted by p, the following two cases are considered.

Case 1: $p = m-1$

Let $\mathcal{H}$ be the set of all $\widetilde{\mathbf{x}}(t,k)$, such that the linear system (2) has a solution with $m-1$ nonzero elements, i.e., where there are $m-1$ active sources at each TF point. Obviously $\mathcal{H}$ is the union of all the C_n^{m-1} subspaces, which are spanned by $m \times (m-1)$ submatrices of the matrix $\mathbf{A}$.

Set $\mathcal{T}$ be the union of all the intersections of any two above subspaces. Clearly, $\mathcal{T}$ has a measure zero in $\mathcal{H}$, which means that any mixture vector $\widetilde{\mathbf{x}}(t,k)$ is in the set $\mathcal{T}$ with probability zero.

For any given mixture TF vector $\widetilde{\mathbf{x}}(t,k) \in \mathcal{H} - \mathcal{T}$, from (11), the constructed vector $\widehat{\mathbf{s}}(t,k)$ satisfies the following equation:

$$\begin{aligned}\mathbf{A}\widehat{\mathbf{s}}(t,k) &= \boldsymbol{\mathcal{A}}_*\mathbf{e}\\ &= \boldsymbol{\mathcal{A}}_*^\dagger \widetilde{\mathbf{x}}(t,k)\\ &= \widetilde{\mathbf{x}}(t,k).\end{aligned} \tag{20}$$

It holds from (2) and (20) that

$$\mathbf{A}[\widehat{\mathbf{s}}(t,k) - \widetilde{\mathbf{s}}(t,k)] = \mathbf{0}. \tag{21}$$

As $\widetilde{\mathbf{x}}(t,k) \notin \mathcal{T}$, $\widetilde{\mathbf{x}}(t,k)$ belongs to only one $(m-1)$-dimensional subspace spanned by $m-1$ columns $\mathbf{a}_{\phi_1}, \mathbf{a}_{\phi_2}, \ldots, \mathbf{a}_{\phi_{m-1}}$ of $\mathbf{A}$. It means that $\widehat{\mathbf{s}}(t,k)$ and $\widetilde{\mathbf{s}}(t,k)$ have $m-1$ nonzero elements in places with indices in $\phi_1, \phi_2, \ldots, \phi_{m-1}$.

From (21), it follows that if $\widehat{\mathbf{s}}(t,k) \neq \widetilde{\mathbf{s}}(t,k)$, the $m-1$ vector columns $\mathbf{a}_{\phi_1}, \mathbf{a}_{\phi_2}, \ldots, \mathbf{a}_{\phi_{m-1}}$ of $\mathbf{A}$ will be linearly dependent, which is a contradiction with Assumption 2.

Case 2: $p < m-1$

For any given mixture TF vector $\widetilde{\mathbf{x}}(t,k)$ produced by p sources, that is the TF vector $\widetilde{\mathbf{s}}(t,k)$ has p nonzeros. Such that $\widetilde{\mathbf{x}}(t,k)$ lies in the space spanned by p columns of the mixing matrix. Based on (11), we get that $\widehat{\mathbf{s}}(t,k)$ also has p nonzeros. We can take out the steering vectors corresponding to p sources, and rewrite (21) as follows:

$$\begin{aligned}&(\mathbf{a}_{\iota_1}, \mathbf{a}_{\iota_2}, \ldots, \mathbf{a}_{\iota_\mathbf{p}})\widehat{\mathbf{s}}_{\iota_1,\ldots,\iota_p}(t,k)\\ &\quad = (\mathbf{a}_{\vartheta_1}, \mathbf{a}_{\vartheta_2}, \ldots, \mathbf{a}_{\vartheta_\mathbf{p}})\widetilde{\mathbf{s}}_{\vartheta_1,\ldots,\vartheta_p}(t,k).\end{aligned} \tag{22}$$

Let $\mathcal{U}$ be the set of all $\widetilde{\mathbf{x}}(t,k)$, such that the linear system (2) has a solution with p nonzero elements. Here, $\mathcal{U}$ is the union of all the C_n^p subspaces, which are produced by $m \times p$ submatrices of the mixing matrix $\mathbf{A}$.

Set $\mathcal{L}$ be the union of all the intersections of the two subspaces spanned by $\mathbf{a}_{\iota_1}, \mathbf{a}_{\iota_2}, \ldots, \mathbf{a}_{\iota_\mathbf{p}}$ and $\mathbf{a}_{\vartheta_1}, \mathbf{a}_{\vartheta_2}, \ldots, \mathbf{a}_{\vartheta_\mathbf{p}}$. As the sources

[4]The blind separation results of our proposed method are available at https://www.dropbox.com/s/vb8d9l69rw93mwh/UBSS.rar?dl=0.

are irrelevant to the mixing matrix, obviously, $\mathcal{L}$ has a measure zero in $\mathcal{U}$, which means that any mixture vector $\widetilde{\mathbf{x}}(t,k)$ is in the set $\mathcal{L}$ with probability zero.

For a vector $\widetilde{\mathbf{x}}(t,k) \in \mathcal{U} - \mathcal{L}$, as $\widetilde{\mathbf{x}}(t,k) \notin \mathcal{L}$, $\widetilde{\mathbf{x}}(t,k)$ can only belongs to the subspace spanned by p columns $\mathbf{a}_{\imath_1}, \mathbf{a}_{\imath_2}, \ldots, \mathbf{a}_{\imath_p}$ or the subspace spanned by p columns $\mathbf{a}_{\vartheta_1}, \mathbf{a}_{\vartheta_2}, \ldots, \mathbf{a}_{\vartheta_p}$.

Let us consider $\widetilde{\mathbf{x}}(t,k)$ belongs to the subspace spanned by p columns $\mathbf{a}_{\imath_1}, \mathbf{a}_{\imath_2}, \ldots, \mathbf{a}_{\imath_p}$. $\widehat{\mathbf{s}}(t,k)$ and $\tilde{\mathbf{s}}(t,k)$ have p nonzero elements in places with indices in $\imath_1, \imath_2, \ldots, \imath_p$. From (21), it follows that if $\widehat{\mathbf{s}}(t,k) \neq \tilde{\mathbf{s}}(t,k)$, the p vector columns $\mathbf{a}_{\imath_1}, \mathbf{a}_{\imath_2}, \ldots, \mathbf{a}_{\imath_p}$ of $\mathbf{A}$ will be linearly dependent, which is a contradiction with Assumption 2.

The same theory proves the case of $\widetilde{\mathbf{x}}(t,k)$ belongs to the subspace spanned by p columns $\mathbf{a}_{\vartheta_1}, \mathbf{a}_{\vartheta_2}, \ldots, \mathbf{a}_{\vartheta_p}$.

Based on the analysis outcomes from the above two cases, we conclude that $\widehat{\mathbf{s}}(t,k)$ is unique and equals to the source signals with probability one. This completes the proof. ■

REFERENCES

[1] A. Belouchrani, K. Abed-Meraim, J.-F. Cardoso, and E. Moulines, "A blind source separation technique using second-order statistics," *IEEE Trans. Signal Process.*, vol. 45, no. 2, pp. 434–444, Feb. 1997.

[2] J. F. Cardoso, "Infomax and maximum likelihood for blind source separation," *IEEE Signal Process. Lett.*, vol. 4, no. 4, pp. 112–114, Apr. 1997.

[3] A. Belouchrani and M. G. Amin, "Blind source separation based on time–frequency signal representations," *IEEE Trans. Signal Process.*, vol. 46, no. 11, pp. 2888–2897, Nov. 1998.

[4] Y. Li, Z. L. Yu, N. Bi, Y. Xu, Z. Gu, and S.-I. Amari, "Sparse representation for brain signal processing: A tutorial on methods and applications," *IEEE Signal Process. Mag.*, vol. 31, no. 3, pp. 96–106, May 2014.

[5] P. Georgiev, F. Theis, and A. Cichocki, "Sparse component analysis and blind source separation of underdetermined mixtures," *IEEE Trans. Neural Netw.*, vol. 16, no. 4, pp. 992–996, Jul. 2005.

[6] P. Bofill and M. Zibulevsky, "Underdetermined blind source separation using sparse representations," *Signal Process.*, vol. 81, no. 11, pp. 2353–2362, 2001.

[7] A. Jourjine, S. Rickard, and O. Yilmaz, "Blind separation of disjoint orthogonal signals: Demixing N sources from 2 mixtures," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process.*, vol. 5. 2000, pp. 2985–2988.

[8] F. Abrard and Y. Deville, "A time–frequency blind signal separation method applicable to underdetermined mixtures of dependent sources," *Signal Process.*, vol. 85, no. 7, pp. 1389–1403, 2005.

[9] V. G. Reju, S. N. Koh, and I. Y. Soon, "An algorithm for mixing matrix estimation in instantaneous blind source separation," *Signal Process.*, vol. 89, no. 9, pp. 1762–1773, 2009.

[10] Y. Li, S.-I. Amari, A. Cichocki, D. W. C. Ho, and S. Xie, "Underdetermined blind source separation based on sparse representation," *IEEE Trans. Signal Process.*, vol. 54, no. 2, pp. 423–437, Feb. 2006.

[11] J.-D. Xu, X.-C. Yu, D. Hu, and L.-B. Zhang, "A fast mixing matrix estimation method in the wavelet domain," *Signal Process.*, vol. 95, pp. 58–66, Feb. 2014.

[12] Y. Luo, W. Wang, J. A. Chambers, S. Lambotharan, and I. Proudler, "Exploitation of source nonstationarity in underdetermined blind source separation with advanced clustering techniques," *IEEE Trans. Signal Process.*, vol. 54, no. 6, pp. 2198–2212, Jun. 2006.

[13] G. R. Naik and W. Wang, *Blind Source Separation: Advances in Theory, Algorithms and Applications*. Springer, 2014.

[14] N. Linh-Trung, A. Belouchrani, K. Abed-Meraim, and B. Boashash, "Separating more sources than sensors using time-frequency distributions," *EURASIP J. Appl. Signal Process.*, vol. 2005, pp. 2828–2847, 2005.

[15] A. Aissa-El-Bey, N. Linh-Trung, K. Abed-Meraim, A. Belouchrani, and Y. Grenier, "Underdetermined blind separation of nondisjoint sources in the time-frequency domain," *IEEE Trans. Signal Process.*, vol. 55, no. 3, pp. 897–907, Mar. 2007.

[16] G. Zhou, Z. Yang, S. Xie, and J.-M. Yang, "Mixing matrix estimation from sparse mixtures with unknown number of sources," *IEEE Trans. Neural Netw.*, vol. 22, no. 2, pp. 211–221, Feb. 2011.

[17] T. Dong, Y. Lei, and J. Yang, "An algorithm for underdetermined mixing matrix estimation," *Neurocomputing*, vol. 104, pp. 26–34, Mar. 2013.

[18] M. A. N. Thirion-Moreau, "Spatial quadratic time-frequency domain methods," in *Handbook of Blind Source Separation, Independent Component Analysis and Applications*, P. Comon and C. Jutten, Eds. Oxford, U.K.: Academic, 2010.

[19] A. Belouchrani, M. G. Amin, N. Thirion-Moreau, and Y. D. Zhang, "Source separation and localization using time-frequency distributions: An overview," *IEEE Signal Process. Mag.*, vol. 30, no. 6, pp. 97–107, Nov. 2013.

[20] N. Linh-Trung, A. Belouchrani, K. Abed-Meraim, and B. Boashash, "Time-frequency signal anlysis and processing: A comprehensive reference," in *Undetermined Blind Sources Separation for FM-Like Signals*, B. Boashash, Ed. Oxford, U.K.: Prentice-Hall, 2003.

[21] E. M. Fadaili, N. T. Moreau, and E. Moreau, "Nonorthogonal joint diagonalization/zero diagonalization for source separation based on time-frequency distributions," *IEEE Trans. Signal Process.*, vol. 55, no. 5, pp. 1673–1687, May 2007.

[22] Y. Z. M. Amin, "Time-frequency signal anlysis and processing: A comprehensive reference," *Spatial Time-Frequency Distributions and their Applications*, B. Boashash, Ed. Oxford, U.K.: Prentice-Hall, 2003.

[23] Y. Xiang, D. Peng, and Z. Yang, *Blind Source Separation: Dependent Component Analysis*. Singapore: Springer, 2015.

[24] D. Peng and Y. Xiang, "Underdetermined blind source separation based on relaxed sparsity condition of sources," *IEEE Trans. Signal Process.*, vol. 57, no. 2, pp. 809–814, Feb. 2009.

[25] D. Peng and Y. Xiang, "Underdetermined blind separation of nonsparse sources using spatial time-frequency distributions," *Digit. Signal Process.*, vol. 20, no. 2, pp. 581–596, 2010.

[26] L. Cohen, *Time-frequency Analysis*, vol. 778. Englewood Cliffs, NJ, USA: Prentice-Hall, 1995.

[27] M. G. Jafari, S. A. Abdallah, M. D. Plumbley, and M. E. Davies, *Sparse Coding for Convolutive Blind Audio Source Separation*. Berlin, Germany: Springer, 2006, pp. 132–139.

[28] J. Bobin, J.-L. Starck, J. Fadili, and Y. Moudden, "Sparsity and morphological diversity in blind source separation," *IEEE Trans. Image Process.*, vol. 16, no. 11, pp. 2662–2674, Nov. 2007.

[29] S. Kim and C. D. Yoo, "Underdetermined blind source separation based on subspace representation," *IEEE Trans. Signal Process.*, vol. 57, no. 7, pp. 2604–2614, Jul. 2009.

[30] D. L. Donoho, "For most large underdetermined systems of linear equations the minimal ℓ_1-norm solution is also the sparsest solution," *Commun. Pure Appl. Math.*, vol. 59, no. 6, pp. 797–829, 2006.

[31] M. S. Asif and J. Romberg, "Sparse recovery of streaming signals using ℓ_1-homotopy," *IEEE Trans. Signal Process.*, vol. 62, no. 16, pp. 4209–4223, Aug. 2014.

[32] X. Peng, Z. Yi, and H. Tang, "Robust subspace clustering via thresholding ridge regression," in *Proc. AAAI Conf. Artif. Intell. (AAAI)*, 2015, pp. 3827–3833.

[33] X. Peng, H. Tang, L. Zhang, Z. Yi, and S. Xiao, "A unified framework for representation-based subspace clustering of out-of-sample and large-scale data," *IEEE Trans. Neural Netw. Learn. Syst.*, to be published.